Analytical Recursive Algorithm for Valuating the Path-dependent Derivatives

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In most cases, the transition density function of an Itô stochastic differential equation is not available in closed-form. Traditionally, one has often resigned in imposing so much structure on the conditional distribution of X so as to remain in a closed-form or quasi closed-form framework. While for the path-dependent derivatives (Asian, Barrier and lookback, etc.), even in the simplest case, no closed-form are available.

In this paper, we developed a recursion algorithm for calculating the moment generation function of certain functionals of Brownian motion when time $\mathbb{T}$ is independently randomly distributed, e.g. exponentially distributed. The integral of the exponential functionals of Brownian motion, which plays a seminal role in path-dependent derivatives pricing, can be obtained by carrying out the real Laplace inversion. We first present a new method to efficiently obtain the Laplace transform of the transition density function for arbitrary diffusion processes of the form

$$X_0 = x_0, \qquad dX_t = \mu(X_t)dt + \sigma(X_t)dW_t.$$

Considering the Laplace transform of this transition density with respect to the time variable, the problem is reduced to the solution of the determination of an ordinary differential equation, where the boundary conditions are such that for x_T (endpoint) approaches infinity, the density equals zero, whereas there is a jump when x_T crosses x_t for the first order derivative. Using Feynman-Kac integration containing a potential and in addition a δ-function perturbation, we construct an exact recursion scheme for the Laplace transform of the transition density and further on for the moment generating function of the functionals of Brownian motion. Finally, we perform the real Laplace inversion based on Gaver functionals with certain nonlinear acceleration sequence transformations to generate very accurate approximations for both short time horizon and long time horizon.

Keywords: continuous-time Markov model, diffusion process, transition density function, Feynman-Kac integration, path-dependent derivatives, real Laplace inversion.

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